## Optimising with more than one variable

## Partial differentiation

When we have a function of more than one variable, we can use partial differentiation to find the rate of change of the function with respect to any of the variables.

Let $\mathrm{F}(\mathrm{X}, \mathrm{Y})$ be a function of two variables. Then the partial differential of F with respect to X , written $\frac{\partial F}{\partial X}$, is obtained simply by differentiating $\mathrm{F}(\mathrm{X}, \mathrm{y})$ with respect to X , holding $Y$ constant, i.e. treating the function as if it were a function only of X , with Y a constant parameter. Similarily, the partial differential of F with respect to $\mathrm{Y}, \frac{\partial F}{\partial Y}$, is obtained by differentiating F with respect to Y , treating X as a constant.

Formally, the partial differentials at the point $(\mathrm{a}, \mathrm{b})$ are given by:

$$
\begin{aligned}
& \frac{\partial F}{\partial X}(a, b)=\operatorname{Lim}_{X \rightarrow a} \frac{F(X, b)-F(a, b)}{X-a} \text { and } \\
& \frac{\partial F}{\partial Y}(a, b)=\operatorname{Lim}_{Y \rightarrow b} \frac{F(a, Y)-F(a, b)}{Y-b}
\end{aligned}
$$

## Example

Consider a Cobb-Douglas production function, given by
$\mathrm{Q}=\mathrm{aK}{ }^{\alpha} \mathrm{L}^{\beta}$, where Q is output, K is capital and L is labour, and $\alpha$ and $\beta$ are constants.

Then $\frac{\partial Q}{\partial K}=a \alpha K^{\alpha-1} L^{\beta}$, and $\frac{\partial Q}{\partial L}=a \beta K^{\alpha} L^{\beta-1}$.

We may of course have functions of any number of variables, for example $\mathrm{F}(\mathrm{X}, \mathrm{Y}, \mathrm{Z})$, a function of 3 variables. We may take partial differentials with respect to any of the variables.

Stationary points of functions of two variables.
Simple optimisation in 2 variables is quite similar to one variable:
A stationary point occurs when all partial differentials are equal to zero. This can be a local maximum, a local minimum, or a saddle point.

In other words, where the function is momentarily flat with respect to changes in any variable.

To find out the nature of the stationary points, we need to look at the second partial derivatives at the stationary point.

Second partial differentials
If $\mathrm{F}(\mathrm{X}, \mathrm{Y})$ is a function of two variables, we may define the second partial derivatives as follows:

The second partial derivative of F wrt $\mathrm{X}, \frac{\partial^{2} F}{\partial X^{2}}=\frac{\partial}{\partial X}\left(\frac{\partial F}{\partial X}\right)$, that is, we differentiate $\frac{\partial F}{\partial X}$ with respect to X .

The second partial derivative of F wrt $\mathrm{Y}, \frac{\partial^{2} F}{\partial Y^{2}}=\frac{\partial}{\partial Y}\left(\frac{\partial F}{\partial Y}\right)$, that is, we differentiate $\frac{\partial F}{\partial Y}$ with respect to Y .

The cross-partial derivative of F with respect to X and $\mathrm{Y}, \frac{\partial^{2} F}{\partial X \partial Y}=\frac{\partial^{2} F}{\partial Y \partial X}=$ $\frac{\partial}{\partial Y}\left(\frac{\partial F}{\partial X}\right)=\frac{\partial}{\partial X}\left(\frac{\partial F}{\partial Y}\right)$. That is, we can either differentiate $\frac{\partial F}{\partial X}$ with respect to Y, or differentiate $\frac{\partial F}{\partial Y}$ with respect to X - the result is always the same.

## Example

Continuing with the CD production function,
$\mathrm{Q}=a \mathrm{~K}^{\alpha} \mathrm{L}^{\beta}$, we had $\frac{\partial Q}{\partial K}=a \alpha K^{\alpha-1} L^{\beta}$ and $\frac{\partial Q}{\partial L}=a \beta K^{\alpha} L^{\beta-1}$
Then, $\frac{\partial^{2} Q}{\partial K^{2}}=a \alpha(\alpha-1) K^{\alpha-2} L^{\beta}, \frac{\partial^{2} Q}{\partial L^{2}}=a \beta(\beta-1) K^{\alpha} L^{\beta-2}$ and
$\frac{\partial^{2} Q}{\partial K \partial L}=a \alpha \beta K^{\alpha-1} L^{\beta-1}$. Note that the last result is the same whichever order we perform the two differentiations in.

## Classifying stationary points of functions of two variables

The nature of a stationary point of a function of two variables depends, unfortunately, on all the second partial derivatives.

Suppose $\mathrm{F}(\mathrm{X}, \mathrm{Y})$ has a stationary point at $(\mathrm{a}, \mathrm{b})$.
Let $\mathrm{A}=\frac{\partial^{2} F}{\partial X^{2}}(\mathrm{a}, \mathrm{b}), \mathrm{B}=\frac{\partial^{2} F}{\partial Y^{2}}(\mathrm{a}, \mathrm{b})$ and $\mathrm{C}=\frac{\partial^{2} F}{\partial X \partial Y}(\mathrm{a}, \mathrm{b})$.
Then ( $\mathrm{a}, \mathrm{b}$ ) is a local maximum if $\mathrm{A}<0$ and $\mathrm{AB}-\mathrm{C}^{2}>0$, a local minimum if $\mathrm{A}>0$ and $\mathrm{AB}-\mathrm{C}^{2}>0$, and a saddle point if $\mathrm{AB}-\mathrm{C}^{2}<0$. (Indeterminate if $\mathrm{AB}-$ $C^{2}=0$ ).

A saddle point will appear to be a local maximum from some directions, and a local minimum from others - like a saddle.

## Example

Let $\mathrm{F}(\mathrm{X}, \mathrm{Y})=\mathrm{X}^{2}-2 \mathrm{Y}^{2}+6 \mathrm{XY}-4 \mathrm{X}+3 \mathrm{Y}$
Then $\frac{\partial F}{\partial X}=2 \mathrm{X}+6 \mathrm{Y}-4$
And $\frac{\partial F}{\partial Y}=-4 \mathrm{Y}+6 \mathrm{X}+3$
Setting these both to zero to find the stationary points gives
$X+3 Y=2$
$6 \mathrm{X}-4 \mathrm{Y}=-3$
Whence $Y=15 / 22$ and $X=-1 / 22$
Now $\frac{\partial^{2} F}{\partial X^{2}}=2, \frac{\partial^{2} F}{\partial Y^{2}}=-4$ and $\frac{\partial^{2} F}{\partial X \partial Y}=6$, so $\frac{\partial^{2} F}{\partial X^{2}} * \frac{\partial^{2} F}{\partial Y^{2}} \leqslant\left(\frac{\partial^{2} F}{\partial X \partial Y}\right)^{2}$ (for all values of X and Y ), which means that the stationary point is a saddle point.

## Convex and concave functions

As in the single variable case, the problem of finding a global maximum or minimum can be more difficult than finding a local optimum. Global optima can occur either at one of the local optima, or at a corner solution.

However, the picture is again clearer for convex and concave functions.
A function $\mathrm{F}(\mathrm{X}, \mathrm{Y})$ is said to be convex over a range of values $\mathbf{A}$ of X and Y if at all points (x,y) in $\mathbf{A}$, we have $\frac{\partial^{2} F}{\partial X^{2}}(x, y) \frac{\partial^{2} F}{\partial Y^{2}}(x, y)-\frac{\partial^{2} F}{\partial X \partial Y}(x, y) \geq 0$ and $\frac{\partial^{2} F}{\partial X^{2}}(x, y)<0$, and concave if $\frac{\partial^{2} F}{\partial X^{2}}(x, y) \frac{\partial^{2} F}{\partial Y^{2}}(x, y)-\frac{\partial^{2} F}{\partial X \partial Y}(x, y) \geq 0$ and $\frac{\partial^{2} F}{\partial X^{2}}(x, y)>0$ for all ( $\mathrm{x}, \mathrm{y}$ ) in A.

These definitions lead to the following results:
If a function $F(X, Y)$ is convex over a region (range of values) $\boldsymbol{A}$, then any local minimum in $\boldsymbol{A}$ is a global minimum for that region. If $F(X, Y)$ is concave on $\mathbf{A}$, then any local maximum is a global maximum on $\mathbf{A}$.

## Non-negativity constraints

If we are seeking to maximise $\mathrm{F}(\mathrm{X}, \mathrm{Y})$ subject to the conditions that $\mathrm{X} \geqslant 0$ and $\mathrm{Y} \geqslant 0$, analogous conditions apply to the 1 variable case:

At the maximum value of $\mathrm{F}(\mathrm{X}, \mathrm{Y})$, we must have $\delta \mathrm{F} / \delta \mathrm{X} \leqslant 0$, with $\delta \mathrm{F} / \delta \mathrm{X}=0$ if $\mathrm{X}>0$, and $\delta \mathrm{F} / \delta \mathrm{Y} \leqslant 0$, with $\delta \mathrm{F} / \delta \mathrm{Y}=0$ if $\mathrm{Y}>0$.

Note that these are not sufficient conditions for a local maximum (we could have a local minimum or a saddle), and certainly not a global maximum, so in general we may have to check a number of different possibilities. While we can check whether we have a local maximum, minimum or saddle using second derivatives for an interior solution (where X and Y are both greater than 0 ), this is not so straightforward where one variable is equal to zero. The conditions for the minimum value are analogous, remembering that minimising $\mathrm{F}(\mathrm{X}, \mathrm{Y})$ is the same as maximising $-\mathrm{F}(\mathrm{X}, \mathrm{Y})$.

A solution to an optimisation problem with non-negativity constraints where one of the variables is equal to zero, is again known as a boundary solution. A solution with all variables strictly greater than zero is an interior solution.

## Functions of several variables

We shall look briefly at the question of finding and classifying stationary points of more than two variables. The process is entirely analogous, but requires the machinery of matrix algebra, which we are not covering here.

Let $\mathrm{F}\left(\mathrm{X}_{1}, \ldots \mathrm{X}_{\mathrm{n}}\right)$ be a function of n variables. A stationary point of F will occur where $\frac{\partial F}{\partial X_{1}}=\ldots=\frac{\partial F}{\partial X_{n}}=0$.

To decide what type of stationary point we have, we need to look at the Hessian matrix of second partial derivatives. This is an $n$ by $n$ array or matrix as follows:

$$
\operatorname{HF}\left(\mathrm{X}_{1}, \ldots, \mathrm{X}_{\mathrm{n}}\right)=\left(\begin{array}{cccc}
\frac{\partial^{2} F}{\partial X_{1}{ }^{2}} & \frac{\partial^{2} F}{\partial X_{1} \partial X_{2}} & \cdots \ldots & \frac{\partial^{2} F}{\partial X_{1} \partial X_{n}} \\
\frac{\partial^{2} F}{\partial X_{1} \partial X_{2}} & \ldots \ldots & \ldots . & \ldots \\
\cdots \ldots \ldots . & & & \ldots \\
\frac{\partial^{2} F}{\partial X_{1} \partial X_{n}} & \frac{\partial^{2} F}{\partial X_{2} \partial X_{n}} & \cdots \ldots & \frac{\partial^{2} F}{\partial X_{n}{ }^{2}}
\end{array}\right)
$$

The type of stationary point will depend on the properties of the Hessian matrix at that point, but the details are beyond the scope of this course.

